Mathematical Visualization to Reinforce the Connection between Linear Algebra and Geometry in Linear Programming

Amal Mostafa
Department of Mathematical Sciences
New Mexico State University
P. O. Box 30001, MSC 3MB
Las Cruces, NM 88003
ammostaf@nmsu.edu

Introduction

Often the courses making the transition to higher mathematics draw on the content of analysis or abstract algebra. Although a rigorous treatment of the concepts of analysis may have great value for students whose first exposure to calculus emphasized graphics and applications, some students will be disappointed with the effort expended to revisit material that they thought they had mastered. For students who consider "introduction to proof" courses to prepare for graduate work in fields such as economics, finance, biology, engineering or statistics, abstract algebra may not be the domain that would best reward their efforts. Introducing proofs in a mathematical domain that draws on geometry and linear, rather than abstract, algebra might appeal to a broader audience, capitalize on the visual and computational skills developed in reform calculus and introduce new mathematical knowledge along with the proofs. Mathematical domains such as differential equations and operations research offer these opportunities.

In this presentation, we illustrate the techniques of proof of the resolution theorem for linear programs in standard form. This theorem gives an algebraic characterization of the geometry of the feasible set of a linear program. The goal of the presentation is to make the geometric content of the theorem and the associated linear algebra more meaningful by creating helpful visualizations of abstract concepts. At each step of the proof, we use visualization to describe geometrically the algebraic techniques in the proof. Since these techniques and their geometric interpretations re-appear in other results on linear programming and in most of the widely used algorithms for solving linear and nonlinear programs, the student who has literally "seen" the proof may recognize it more readily when s/he encounters it again. Mathematical visualization of results in linear programming offers an added motivation to appreciate linear algebra because computers can only handle the geometry of linear programming by means of numeric computation. As a result, the proofs of major results in linear programming have an immediate application because they provide algorithmic procedures by which a computer can solve the large models characteristic of applications.

Resolution Theorem

Standard texts in Linear Programming, such as Murty (1983), Nash and Sofer (1996), use the production procedure to give a complete characterization of the feasible set of a linear program in standard form. This characterization is given in the following theorem.

Theorem: *Consider the set* $S = \{ x\colon\ Ax = b, x \geq 0 \}$*, representing the feasible region for a feasible linear program in standard form. Let* $V = \{ v_1, v_2, \dots, v_k \}$ *be the set of basic feasible solutions of* S*. Then* $x = \sum \lambda_i v_i + y$ *where*

$$\sum \lambda_i = 1 \text{ and } \lambda_i \geq 0, \quad i = 1, \dots, k,$$
$$Ay = 0 \text{ and } y \geq 0.$$

Proof: Choose a feasible solution x. If $x = 0$, then x is a basic feasible solution and $y = 0$.

If $x \neq 0$, consider the set of columns $\{ A_{\cdot j}\colon x_j > 0 \}$. If these vectors are linearly independent, we have a basic feasible solution. If not, there are real numbers $\alpha_1, \dots, \alpha_n$ for which

$$A\alpha = 0$$
$$\alpha_j = 0, \text{ when } x_j = 0$$
$$\text{Min } \alpha_j < 0.$$

Define a strictly positive number θ by

$$\theta = \min \{ -x_j / \alpha_j\colon\ \alpha_j < 0 \}.$$

Construct the vector $x(\theta) = x + \theta\alpha$, then $x(\theta)$ is a feasible solution in which at most (k-1) of the variables are positive.

If Max $\alpha_j \leq 0$ then let $y = -\alpha$; notice that $y \geq 0$ and $Ay = 0$ and that

$$x = (x + \theta\alpha) + y.$$

Otherwise, define the strictly positive number θ' by

$$\theta' = \min \{ x_j / \alpha_j\colon\ \alpha_j > 0 \}.$$

Write the feasible solution x as

$$x = [\theta'(x + \theta\alpha) + \theta(x - \theta'\alpha)] / (\theta + \theta').$$

Now we have written x as a convex combination of feasible solutions, each having at least one more coordinate equal to zero than before. We see that in at most n iterations, we will have achieved the required resolution. |

Now, we want to demonstrate an example of the algorithm used to prove the resolution theorem. We recall Polya's comment, " The beginner, however, should construct many figures as exactly as he can in order to acquire a good experimental basis." (Polya, 1988, P. 105)

Example: We consider the following set of constraints.

$$\begin{aligned} -3x_1 + 2x_2 &\leq 8 \\ x_1 + 2x_2 &\leq 16 \\ x_1 \qquad &\leq 6 \\ x_1 - x_2 &\leq 4 \\ x_1\ ,\ x_2 &\geq 0 \end{aligned}$$

The feasible region of these constraints S is shown in Figure 1.

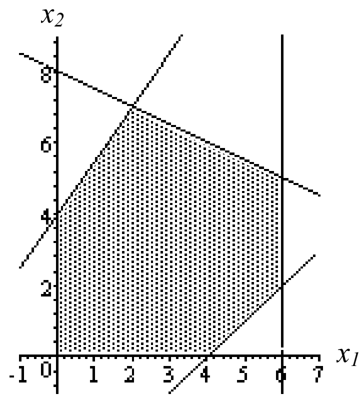

Figure 1. Bounded feasible region represented by rows of a constraint matrix.

The constraints of the example are written in standard form by adding one nonnegative variable for each constraint and then converting the inequalities to equations. The feasible set becomes a subset of R^6 and the constraint matrix A for the linear program in standard form is

$$A= \begin{pmatrix} -3 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In order to find a basic feasible solution we will use the reduction procedure, let us arbitrarily choose the feasible point in standard form $x = (4, 2, 16, 8, 2, 2)$. We test the columns of A for independence by augmenting A^T with identity matrix I_6 and perform pivot operations until we obtain the first zero-row in A^T, from which the vector α can be constructed.

I_1	I_2	I_3	I_4	I_5	I_6	$A^T{}_{.1}$	$A^T{}_{.2}$	$A^T{}_{.3}$	$A^T{}_{.4}$
1	0	0	0	0	0	-3	1	1	1
0	1	0	0	0	0	2	2	0	-1
0	0	1	0	0	0	1	0	0	0
0	0	0	1	0	0	0	1	0	0
0	0	0	0	1	0	0	0	1	0
0	0	0	0	0	1	0	0	0	1

The result of testing the columns for linear dependence is shown next.

α_1	α_2	α_3	α_4	α_5	α_6				
0	0	1	0	0	0	1	0	0	0
0	0	0	1	0	0	0	1	0	0
1	1	1	-3	0	0	0	0	1	0
0	-1	2	2	0	0	0	0	0	1
-1	-1	-1	3	1	0	0	0	0	0
0	1	-2	-2	0	1	0	0	0	0

We choose $\alpha = (-1, -1, -1, 3, 1, 0)$. Since $-A_{.1} - A_{.2} - A_{.3} + 3A_{.4} + A_{.5} = 0$, we verify $A\alpha = 0$. Here θ

$= \min \{ -x_j / \alpha_j\colon\ \alpha_j < 0 \} = \min \{ -4/-1, -2/-1, -16/-1 \} = 2$.

So we construct a new feasible solution $x(\theta) = x + \theta\alpha = (2, 0, 14, 14, 4, 2)$ which lies at the boundary of the feasible set (see Figure 2). Notice that $x(\theta)$ has five positive coordinates, where x had six.

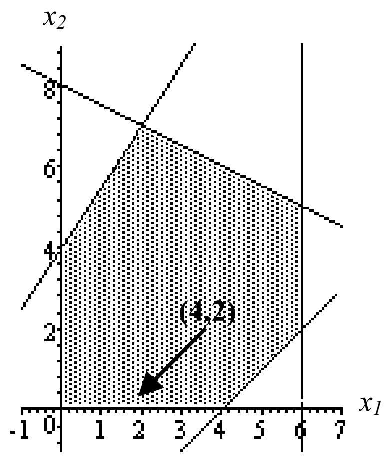

Figure 2. Constructing a boundary feasible solution from the initial feasible solution.

Now, we follow the procedure of the resolution theorem to also consider a step from x in the direction $-\alpha$. We apply a ratio test to compute $\theta' = \min \{ x_j / \alpha_j\colon\ \alpha_j > 0 \} = \min \{ 8/3, 2/1 \}$ $= 2$. Notice that the ratio θ' represents the size of the largest step that can be taken from x in the direction of $-\alpha$ without leaving the feasible set. Then x can be written as a convex combination of feasible solutions, each having one more coordinate equal to zero than before:

$$x = [\theta'(x + \theta\alpha) + \theta(x - \theta'\alpha)] / (\theta + \theta')$$
$$= 1/2(2, 0, 14, 14, 4, 2) + 1/2(6, 4, 18, 2, 0, 2).$$

This first iteration of the algorithm used in the resolution theorem can be seen in Figure 3.

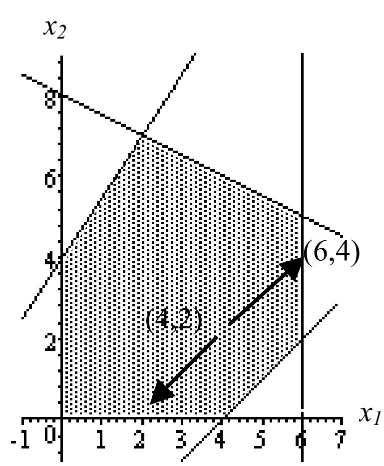

Figure 3. Constructing two reduced feasible solutions from an initial feasible solution.

We still need to reduce the number of positive entries in each of the new feasible solutions. We test the set of columns selected by the positive coordinates of the feasible point (2, 0, 14, 14, 4, 2) for linearly independence. We compute
α = (-1, 0, -3, 1, 1, 1) and we use ratio tests to compute $\theta = 2$ and $\theta' = 2$.
Now we can present (2, 0, 14, 14, 4, 2) as

$$(2, 0, 14, 14, 4, 2) = 1/2(0, 0, 8, 16, 6, 4) + 1/2(4, 0, 20, 12, 2, 0).$$

It can easily be checked that the feasible solutions appearing on the right side of this equation are both basic feasible solutions. This can be seen in Figure 4.

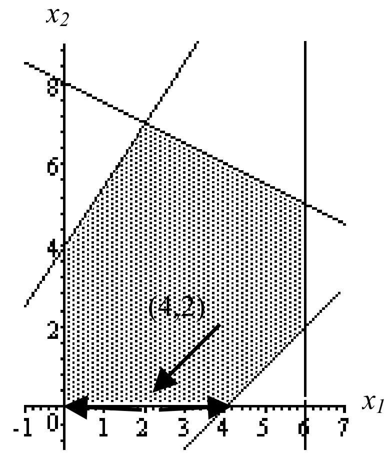

Figure 4. Constructing two basic feasible solutions
from the boundary feasible solution.

Now we test the columns of the constraint matrix selected by the positive coordinates of the feasible point (6, 4, 18, 2, 0, 2) for linearly independence. We compute α = (0, 1, -2, -2, 0, 1), $\theta = 1$, and $\theta' = 2$. Now we can present
(6, 4, 18, 2, 0, 2) as a convex combination of feasible solutions each having one more coordinate equal to zero than before:

$$(6, 4, 18, 2, 0, 2) = 2/3(6, 5, 16, 0, 0, 3) + 1/3(6, 2, 22, 6, 0, 0).$$

It is easily checked that the feasible solutions on the right side of this equation are both basic feasible solutions, as suggested by Figure 5.

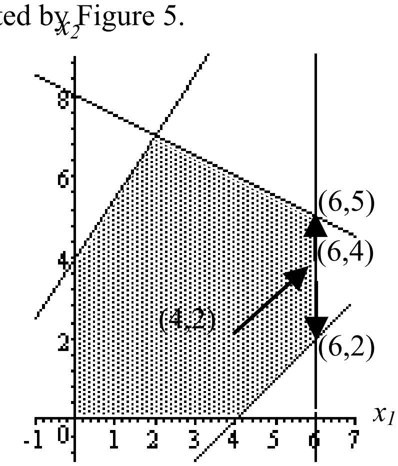

Figure 5. Constructing two basic feasible solutions from the boundary feasible solution.

Finally, we can present the initial feasible point $x = (4, 2, 16, 8, 2, 2)$ as a convex combination of basic feasible solutions that fit the description of the Resolution Theorem:

$$\begin{aligned}(4, 2, 16, 8, 2, 2) = \ & 1/4\,(0, 0, 8, 16, 6, 4) \ + \\ & 1/4\,(4, 0, 20, 12, 2, 0) \ + \\ & 1/6\,(6, 2, 22, 6, 0, 0) \ + \\ & 1/3\,(6, 5, 16, 0, 0, 3)\,.\end{aligned}$$

The four basic feasible solutions are shown in Figure 6.

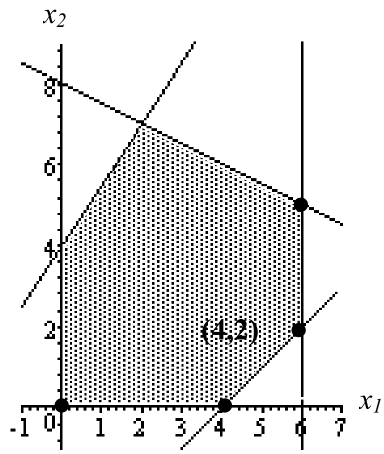

Figure 6. The initial feasible solution can be expressed as a convex combination of basic feasible solutions.

Bibliography

K. G. Murty, *Operations Research*: *Deterministic Optimization Models*, Prentice-Hall, Englewood Cliffs, NJ, 1983.

S. G. Nash, and A. Sofer, *Linear and Nonlinear Programming*, McGraw-Hill, New York, 1996.

G. Polya, *How to Solve It*: *A New Aspect of Mathematical Method*, Princeton University Press, NJ, 1988.